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PATHWAY FRACTIONAL INTEGRAL OPERATOR INVOLVING CERTAIN SPECIAL FUNCTIONS

Vinod Gill[1*], Kanak Modi[1]

[1]Department of Mathematics, Amity University, Rajasthan, Jaipur, India

Abstract. In this paper, we have established certain theorems on pathway fractional integral operator involving products of a general class of polynomials with two H-functions. Certain interesting special cases are also recorded citing known and unknown special cases both.

Keywords: *Pathway fractional integral operator, pathway model, Fox's H-function, a general class of polynomials, Mittag-Leffler function, generalized hypergeometric functions.*

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Corresponding Author: *Dr. Vinod Gill, Department of Mathematics, Amity University Rajasthan Jaipur-303002, India, e-mail: vinod.gill08@gmail.com*

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1. Introduction

The Pathway fractional integral operator introduced by Nair [12] is defined as follows:

$$\left(P_{0+}^{(\eta,\alpha)} f\right)(x) = x^{\eta} \int_{0}^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}} f(t)dt, \tag{1}$$

where $f(x) \in L(a,b)$, $\eta \in C$, $\mathrm{Re}(\eta) > 0$, $a > 0$ and pathway "parameter" $\alpha < 1$.

The pathway model is introduced by Mathai [7] and studied by Mathai and Haubold ([8],[9]). The pathway model for scalar random variables, for real scalar α, is denoted by the following probability density function (p.d.f.).

$$f(x) = c\,|x|^{\gamma-1}[1-a(1-\alpha)|x|^{\delta}]^{\frac{\beta}{1-\alpha}}, \tag{2}$$

$-\infty < x < \infty$, $\delta > 0$, $\beta \geq 0$, $\{1-a(1-\alpha)|x|^{\delta}\} > 0$, $\gamma > 0$, where c is the normalizing constant and α is called the pathway parameter. For real α, the normalizing constant is as follows:

$$c = \frac{1}{2}\frac{\delta[a(1-\alpha)]^{\frac{\gamma}{\delta}}\Gamma\left(\frac{\gamma}{\delta}+\frac{\beta}{1-\alpha}+1\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{1-\alpha}+1\right)}, \alpha < 1 \tag{3}$$

$$=\frac{1}{2}\frac{\delta[a(1-\alpha)]^{\frac{\gamma}{\delta}}\Gamma\left(\frac{\beta}{1-\alpha}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{1-\alpha}-\frac{\gamma}{\delta}\right)}, \text{ for } \frac{1}{\alpha-1}-\frac{\gamma}{\delta}>0,\ \alpha>1 \tag{4}$$

$$=\frac{1}{2}\frac{\delta(a\beta)^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)} \text{ for } \alpha\to 1. \tag{5}$$

For $\alpha<1$, it is a finite range density with $\{1-a(1-\alpha)|x|^{\delta}\}>0$ and (2) remains in the extended generalize type – 1 beta family. The pathway density in (2), for $\alpha<1$, includes the extended type – 1 beta density, the triangular density, the uniform density and many other p.d.f.

For $\alpha<1$, we have

$$f(x)=c\,|x|^{\gamma-1}[1+a(\alpha-1)|x|^{\delta}]^{\frac{\beta}{\alpha-1}}, \tag{6}$$

$-\infty<\mathrm{x}<\infty$, $\delta>0$, $\beta\geq 0$, $\alpha>1$, which is extended generalized type-2 beta model for real x. It includes the type- 2 beta density, the F-density, the student $-t$ density, The Cauchy density and many more. Here it is considered only the case of pathway parameter $\alpha<1$. For $\alpha\to 1$, (2) and (6) take the exponential form, since

$$\begin{aligned}&\lim_{\alpha\to 1} c\,|x|^{\gamma-1}[1-a(1-\alpha)|x|^{\delta}]^{\frac{\eta}{1-\alpha}}\\ &=\lim_{\alpha\to 1} c\,|x|^{\gamma-1}[1+a(\alpha-1)|x|^{\delta}]^{\frac{\eta}{\alpha-1}}=c\,|x|^{\gamma-1}e^{-a\eta|x|^{\delta}}\end{aligned} \tag{7}$$

This includes the generalized Gamma-, the Weibuli-, the Chi-square, the Laplace-, and the Maxwell-Boltzmann and other related densities. Therefore, the operator introduced in this paper can be related and applicable to a wide variety of statistical densities.

For more details on the pathway model, the reader is referred to the papers of Mathai and Haubold ([8],[9]). It is seen that the pathway fractional integral operator (1), based on the pathway model of Mathai and Haubold, and using the pathway parameter *α,* can lead to other interesting examples of fractional calculus operators, related to some probability density functions and applications in statistics.

The H-function, introduced by Fox ([4], p.408) is represented and defined in the following manner:

$$H^{m,n}_{p,q}[z]=H^{\mathrm{m,n}}_{\mathrm{p,q}}\left[\mathrm{z}\left|\begin{matrix}(\mathrm{u_p},\mathrm{U_p})\\(\mathrm{v_q},\mathrm{V_q})\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_{\mathrm{L}}\Phi(\xi)\,z^{\xi}\,d\xi,\ \ \omega=\sqrt{-1} \tag{8}$$

z may be real or complex but not equal to zero and an empty product is interpreted as unity; the integers m, n, p, q are such that $0\leq n\leq p$, $1\leq m\leq q$, $U_1,...,U_p$ and $V_1,...,V_q$ are positive integers; $u_1,...,u_p$ and $v_1,...,v_q$ are complex numbers and such that no poles of $\Gamma(v_i - V_i\xi)(i = 1,...,m)$ coincide with any poles of $\Gamma(1-u_i + U_i\xi)(i = 1,...,n)$,

$$\Phi(\xi) = \frac{\prod_{i=1}^{m} \Gamma(v_i - V_i\xi) \prod_{i=1}^{n} \Gamma(1-u_i + U_i\xi)}{\prod_{i=m+1}^{q} \Gamma(1-v_i + V_i\xi) \prod_{i=n+1}^{p} \Gamma(u_i - U_i\xi)} \tag{9}$$

i.e.

$$U_i(v_j + h) \neq (u_i - k - 1)V_j \tag{10}$$

$(h,k = 0,1...., j;\ j = 1,...,m;\ i = 1,...,n).$

The contour L runs from c–i∞ to c+i∞ (c is real) in such a manner that the points

$$\xi = \frac{v_j + h}{V_j},\ j = 1,...,m\ ;\ h = 0,1,2,...,$$

which are the poles of $\Gamma(v_i - V_i\xi)(i = 1,...,m)$ lie to the right and the points given by

$$\xi = \frac{u_i - k - 1}{U_i}, i = 1,...,m\ ;\ k = 0,1,2,...$$

which are the poles of $\Gamma(1-u_i + U_i\xi)(i = 1,...,n)$ lie to the left of the contour L. Such a contour is possible on account of (9). These assumptions will be retained through.

$$(u_p, U_p) \text{ stands for } (u_1, U_1),...,(u_p, U_p).$$

Braaksma ([1], p.278, Th.1) has shown the H-function makes sense and defines an analytic function of z in the following cases:

(i) $$\delta = \left(\sum_{1}^{q} (V_i) - \sum_{1}^{p} (U_i) \right) > 0,\ z \neq 0 \tag{11}$$

(ii) $$\delta = 0, \text{ i.e. } \sum_{1}^{q} V_i = \sum_{1}^{p} U_i \text{ and } 0 < |z| < D^{-1} \tag{12}$$

where

$$D = \prod_{i=1}^{p} (U_i)^{U_i} \prod_{i=1}^{q} (V_i)^{-V_i}.$$

In these two cases, we have, Braaksma ([1], p.279, eqn. (6.5.1),

$$H_{p,q}^{m,n}[z] = 0\,(|z|^{\lambda})\ , \text{ for small } z \tag{13}$$

where $\lambda = \min.\ \mathrm{Re}\left(\frac{v_i}{V_i}\right)(i = 1,...,m).$

The integral in (8) converges absolutely if ([1], p.50)

(iii) $$T = \sum_{1}^{n} U_i - \sum_{n+1}^{p} U_i + \sum_{1}^{m} V_i - \sum_{m+1}^{q} V_i > 0, \tag{14}$$

and

(iv) $\quad |\arg\ z| < \frac{1}{2}\pi T.$ (15)

In case $T = 0$ and δ > 0, the integral (8) does not converge for z. In this case the integral (8) converges absolutely, if c is so chosen that $z > 0$, ([18], p.50)

(v) $$c\left\{\sum_{1}^{q} V_i - \sum_{1}^{p} U_i\right\} > \operatorname{Re}\left\{\sum_{1}^{q} v_i - \sum_{1}^{p} u_i\right\} - \frac{1}{2}(q-p) + 1. \quad (16)$$

For more convergence conditions, existence of various contours L and other properties, see Mathai and Saxena [10], Kilbas-Saigo-Saxena [5], Kilbas-Srivastava-Trujillo[6], etc. The importance of Fox's H-function lies in the fact that almost all the elementary and special functions in the literature follow as its special cases. These special functions appear in various problems arising in theoretical and applied branches of mathematics, statistics, physics, engineering and other areas.

The series representation of the H-function has been studied in [17]

$$H_{P,Q}^{M,N}\left[z\left|\begin{smallmatrix}(e_P,E_P)\\(f_Q,F_Q)\end{smallmatrix}\right.\right] = \sum_{h=1}^{N}\sum_{v=0}^{\infty}\frac{(-1)^{v}\chi(\xi)}{v\,!\,E_h}\left(\frac{1}{z}\right)^{\xi}, \quad (17)$$

where $\xi = \frac{e_h - 1 - v}{E_h}$ and $(h = 1,...,N)$

and

$$\chi(\xi) = \frac{\prod_{j=1}^{M}\Gamma(f_j + F_j\xi)\prod_{\substack{j=1\\ j\neq h}}^{N}\Gamma(1 - e_j - E_j\xi)}{\prod_{j=M+1}^{Q}\Gamma(1 - f_j - F_j\xi)\prod_{j=N+1}^{P}\Gamma(e_j + \xi E_j)} \quad (18)$$

the convergence conditions and other details are given in [17].

Srivastava ([14], p.1,eqn. (1)] introduced the general class of polynomials

$$S_n^m[x] = \sum_{\ell=0}^{[n/m]}\frac{(-n)_{m\ell}}{\ell!}A_{n,\ell}, \qquad \ell = 0,1,2,... \quad (19)$$

where m is an arbitrary positive integer and the coefficients $A_{n,l}\,(n,l \geq 0)$ are arbitrary constants, real or complex. By suitably specializing the coefficients $A_{n,\ell}$ occurring in (19) can be reduced to the classical orthogonal polynomials and the generalized hypergeometric polynomials. For further details, we can refer to Srivastava and Singh [15], Srivastava [14], Srivastava and Pathan [16] and Erdélyi [3] .

The following result will be required in our sequel

$$\int_0^{\frac{x}{a(1-\alpha)}}\left[1 - \frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}} t^{\beta-1}\,dt = \frac{x^{\eta+\beta}\,\Gamma(\beta)\,\Gamma\left(1 + \frac{\eta}{1-\alpha}\right)}{a(1-\alpha)]^{\beta}\,\Gamma\left(\frac{\eta}{1-\alpha} + \beta + 1\right)}, \quad (20)$$

$\alpha < 1, Re(\eta) > 0,\ \mathrm{Re}\,(\beta) > 0.$

2. The main theorem

Theorem 1. Let $\eta, \rho \in \mathrm{C}, \alpha < 1, b \in R, C \in R, \beta' > 0$ and

(i) $\mathrm{Re}(\beta) > 0,\ \mathrm{Re}\,(\delta) > 0, \mathrm{Re}\,(\rho) > 0,$

(ii) $\mathrm{Re}\left(1 + \frac{\eta}{1-\alpha}\right) > 0,$

(iii) $\mathrm{Re}\left(\rho + \delta \frac{f_j}{F_j}\right) > 0,$

(iv) $\mathrm{Re}\left(\rho + \beta \frac{b_j'}{\beta_j}\right) > 0,\ j = 1,...,M;\ j' = 1,...,m$

(v) m' is an arbitrary positive integer and the coefficients $A_{n',k}(n',k \geq 0)$ are arbitrary constants, real or complex. Then

$$P_{0+}^{(\eta\eta,\alpha}\left\{t^{\rho-1} S_{n'}^{m'}[dt^{-\beta'}] H_{P,Q}^{M,N}\left[c\, t^{\delta}\middle|\begin{smallmatrix}(e_P,E_P)\\(f_Q,F_Q)\end{smallmatrix}\right] H_{p,q}^{m,n}\left[b\, t^{\beta}\middle|\begin{smallmatrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{smallmatrix}\right]\right\}$$

$$= \sum_{k=0}^{[n'/m']} \frac{(-n')_{m'k}}{k\,!} A_{n',k} \frac{d^k\, x^{\eta+\rho-\beta'k}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho-\beta'k}} H_{P,Q}^{M,N}\left[\frac{c\, x^{\delta}}{a(1-\alpha)^{\delta}}\middle|\begin{smallmatrix}(e_P,E_P)\\(f_Q,F_Q)\end{smallmatrix}\right]$$

$$.\ H_{p+1,q+1}^{m,\, n+1}\left[\frac{b\, x^{\beta}}{a(1-\alpha)^{\beta}}\middle|\begin{smallmatrix}(1-\rho+\delta'\xi+\beta'k,\beta),(a_p,\alpha_p)\\(b_q,\beta_q),\left(-\rho+\delta'\xi+\beta'k-\frac{\eta}{1-\alpha},\beta\right)\end{smallmatrix}\right]. \tag{21}$$

Proof. Making use of (19), (17), (8) and (1) and applying to a known result (20), after a little simplification, we arrive at the desired result (21).

Theorem 2. Let $\eta > 0, \gamma > 0, \delta > 0, b \in R,\ c \in R,\ \beta > 0,\ \alpha < 1$ and

(i) $\mathrm{Re}(\rho) > 0, \mathrm{Re}(\eta) > 0,\ \mathrm{Re}(\gamma) > 0,$

(ii) $\mathrm{Re}\left(1 + \frac{\eta}{1-\alpha}\right) > \max\,[0, -\mathrm{Re}(\rho)],$

(iii) $\mathrm{Re}\left(\rho + \delta \frac{\mathrm{f_j}}{\mathrm{F_j}}\right) > 0,\ j = 1,...,M,$

(iv) m' is an arbitrary positive integer and the coefficients $A_{n',k}(n',k \geq 0)$ are arbitrary constants, real or complex.

(v) $E_{\beta,\rho}^{\gamma}(x)$ is the generalized Mittag-Leffler function [13]. Then

$$P_{0+}^{(\eta,\alpha)}\left\{t^{\rho-1} S_{n'}^{m'}[dt^{-\beta'}] E_{\beta,\rho}^{\gamma}(bt^{\beta})\, H_{P,Q}^{M,N}\left[c\, t^{\delta}\middle|\begin{smallmatrix}(e_P,E_P)\\(f_Q,F_Q)\end{smallmatrix}\right]\right\}$$

$$= \sum_{k=0}^{[n'/m']} \frac{(-n')_{m'k}}{k\,!} A_{n',k} \frac{d^k\, x^{\eta+\rho-\beta'k}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\Gamma(\gamma)\,[a(1-\alpha)]^{\rho-\beta'k}}$$

$$\cdot\ {}_2\psi_2\left[\frac{b\,x^{\beta}}{[a(1-\alpha)]^{\beta}}\left|\begin{matrix}(\rho-\delta\xi-\beta'\mathrm{k},\beta),(\gamma,1)\\(\rho,\beta),\left(1+\rho+\frac{\eta}{1-\alpha}-\delta\xi-\beta'\mathrm{k},\beta\right)\end{matrix}\right.\right]$$

$$.\,H_{P,Q}^{M,N}\left[\frac{c\,x^{\delta}}{[a(1-\alpha)]^{\delta}}\left|\begin{matrix}(\mathrm{e_P},\mathrm{E_P})\\(\mathrm{f_Q},\mathrm{F_Q})\end{matrix}\right.\right]. \tag{22}$$

where $E_{\beta,\rho}^{\gamma}(b)$ is the generalized Mittag-Leffler function (see [11], [13]).

Proof. In the theorem 1, putting

$$m=1, n=1,\ p=1, q=2, b_1=0, \beta_1=1, b_2=1-\rho, \beta_2=\beta, \alpha_1=1-\gamma$$

and $\alpha_1=1$, after a little simplification, we have the desired result.

Theorem 3. Let $\eta,\gamma,v\in C,\ c\in R,\ \delta>0,\ \alpha<1,\ d=1$ and

(i) $\operatorname{Re}\left(1+\frac{\eta}{1-\alpha}\right)>0, \operatorname{Re}(\eta)>0,$

(ii) $\operatorname{Re}(\gamma+v)>0,$

(iii) $\operatorname{Re}\left(\gamma+\delta\frac{f_j}{F_j}\right)>0,\ j = 1,\ldots,M,$

(v) m' is an arbitrary positive integer and the coefficients $A_{n',k}(n',k\geq 0)$ are arbitrary constants, real or complex. Then

$$P_{0+}^{(\eta,\alpha)}\left\{\left(\frac{t}{2}\right)^{\gamma-1}H_{P,Q}^{M,N}\left[c\left(\frac{t}{2}\right)^{\delta}\left|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right.\right]S_{n'}^{m'}\left[d\left(\frac{t}{2}\right)^{-\beta'}\right]J_v(t)\right\}$$

$$=\sum_{k=0}^{[n'/m']}\frac{(-n')_{m'k}d^k}{k!}A_{n',k}\frac{x^{\eta+v+\gamma-\beta'k}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\gamma+v-\beta'k}\,2^{\gamma+v+\eta-\beta'k}}H_{P,Q}^{M,N}\left[\frac{c\,x^{\delta}}{[2a(1-\alpha)]^{\delta}}\left|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right.\right]$$

$$\cdot\ {}_1\psi_2\left[-\frac{x^2}{4a^2(1-\alpha)^2}\left|\begin{matrix}(\gamma+v-\delta\xi-\beta'k,2)\\(v+1,1),(1+v+\gamma-\delta\xi-\beta'k+\frac{\eta}{1-\alpha},2\end{matrix}\right.\right]. \tag{23}$$

Here ${}_p\psi_q$ denotes the generalized Wright hypergeometric function [18].

Proof. The result in (23) can be obtained by putting in the theorem 1, $m=1, n=0, q=2, b_1=0,\ \beta_1=1,\ b_2=-v, \beta_2=1, e=\gamma+v,\ b'=1, \beta=2$ and replacing t by $\frac{t}{2}$, after a little simplification, we have the required result.

3. Special cases

(A) If we put $m'=2, A_{n',k}=(-1)^{\mathrm{k}}$ and $d=1$ in (21), we have the following theorem

Theorem 1(a). Let $\eta,\rho\in C,\ \alpha<1, b\ \in R,\ c\ \in R, \beta' > 0,$ and

(i) $\operatorname{Re}(\beta)>0, \operatorname{Re}(\delta)>0, \operatorname{Re}(\rho)>0,$

(ii) $\mathrm{Re}\left(1+\frac{\eta}{1-\alpha}\right) > 0,$

(iii) $\mathrm{Re}\left(\rho+\delta\frac{\mathrm{f_j}}{\mathrm{F_j}}\right) > 0,$

(iv) $\mathrm{Re}\left(\rho+\beta\frac{\mathrm{b'_j}}{\beta_{\mathrm{j}}}\right) > 0,\ j = 1,...,M\ ;\ j' = 1,...,m$. Then

$$P_{0+}^{(\eta,\alpha)}\left\{t^{\rho-1} H_{n'}\left(\frac{1}{2\sqrt{t^{-\beta'}}}\right) t^{-n'\beta'\beta 2}\ H_{P,Q}^{M,N}\left[c\ t^{\delta}\left|\begin{smallmatrix}(e_P,E_P)\\(f_Q,F_Q)\end{smallmatrix}\right.\right] H_{p,q}^{m,n}\left[b\ t^{\beta}\left|\begin{smallmatrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{smallmatrix}\right.\right]\right\}$$

$$=\sum_{k=0}^{[n'/2]} \frac{(-n')_{2k}(-1)^k d^k\ x^{\eta+\rho-\beta' k}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{k\ !\ [a(1-\alpha)]^{\rho-\beta'\mathrm{k}}}\ H_{P,Q}^{M,N}\left[\frac{c\,x^{\delta}}{\mathrm{a}(1-\alpha)^{\delta}}\left|\begin{smallmatrix}(e_P,E_P)\\(f_Q,F_Q)\end{smallmatrix}\right.\right]$$

$$\cdot\ H_{p+1,q+1}^{m,\ n+1}\left[\frac{b\,x^{\beta}}{a(1-\alpha)^{\beta}}\left|\begin{smallmatrix}(1-\rho+\delta'\xi+\beta' k,\beta),(a_p,\alpha_p)\\(b_q,\beta_q),\left(-\rho+\delta'\xi+\beta' k-\frac{\eta}{1-\alpha},\beta\right)\end{smallmatrix}\right.\right]. \tag{24}$$

(B) On taking $m'=1$ and $A_{n',k}=\binom{n'+f}{n'}\frac{(f+g+n'+1)_k}{(f+1)_k}$ in (21), we get the following theorem

Theorem 2(a). Let $\eta>0,\ \gamma>0, \delta>0,\ b\in R,\ c\in R,\ \beta>0,\ \alpha<1$ and

(i) $\mathrm{Re}(\rho)>0, \mathrm{Re}(\eta) > 0,\ \mathrm{Re}(\gamma) > 0,$

(ii) $\mathrm{Re}\left(1+\frac{\eta}{1-\alpha}\right) > \max\,[0, -\mathrm{Re}(\rho)],$

(iii) $\mathrm{Re}\left(\rho+\delta\frac{f_j}{F_j}\right) > 0,\ j = 1,...,M,$

(v) $E_{\beta,\rho}^{\gamma}(x)$ is the generalized Mittag-Leffler function [13]. Then

$$P_{0+}^{(\eta,\alpha)}\left\{t^{\rho-1} P_{\mathrm{n'}}^{(f,g)}(1-2t^{-\beta'})\,E_{\beta,\rho}^{\gamma}(bt^{\beta})\ H_{P,Q}^{M,N}\left[c\ t^{\delta}\left|\begin{smallmatrix}(e_P,E_P)\\(f_Q,F_Q)\end{smallmatrix}\right.\right]\right\}$$

$$=\sum_{\mathrm{k=0}}^{\mathrm{n'}} (-1)^{\mathrm{k}}\binom{n'+f}{n'-k}\binom{n'+f+g+k}{k}\frac{d^k\ x^{\eta+\rho-\beta' k}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\Gamma(\gamma)\,[a(1-\alpha)]^{\rho-\beta'\mathrm{k}}}$$

$$\cdot\ {}_2\psi_2\left[\frac{b\,x^{\beta}}{[a(1-\alpha)]^{\beta}}\left|\begin{smallmatrix}(1-\delta\xi-\beta'\mathrm{k},\beta),(\gamma,1)\\(\rho,\beta),\left(1+\rho+\frac{\eta}{1-\alpha}-\delta\xi-\beta'\mathrm{k},\beta\right)\end{smallmatrix}\right.\right].H_{P,Q}^{M,N}\left[\frac{c\,x^{\delta}}{[a(1-\alpha)]^{\delta}}\left|\begin{smallmatrix}(e_P,E_P)\\(f_Q,F_Q)\end{smallmatrix}\right.\right]. \tag{25}$$

(C) Letting m = 1 and $A_{n',k}=\binom{n'+f}{n'}\frac{1}{(f+1)_{\mathrm{n'}}}$, in (21), we get the following theorem

Theorem 3(a). Let $\eta,\gamma,\nu\in C,\ c\in R, \delta>0,\ \alpha<1$ and

(i) $\quad \mathrm{Re}(\eta) > 0,\ \mathrm{Re}\left(1+\dfrac{\eta}{1-\alpha}\right) > 0,$

(ii) $\quad \mathrm{Re}\left(\gamma+\delta\dfrac{f_j}{F_j}\right) > 0,\ j = 1,\ldots,M,$

(iii) $\quad \mathrm{Re}\,(\gamma+v) > 0.$

Then

$$P_{0+}^{(\eta,\alpha)}\left\{\left(\frac{t}{2}\right)^{\gamma-1} H_{P,Q}^{M,N}\left[c\left(\frac{t}{2}\right)^{\delta}\Bigg|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right] L_{n'}^{(f)}\left[\left(\frac{t}{2}\right)^{-\beta'}\right] J_{v}(t)\right\}$$

$$=\sum_{\mathrm{k}=0}^{\mathrm{n'}} \frac{(-1)^{\mathrm{k}}}{k\,!}\binom{n'+f}{n'-k} \frac{x^{\eta+\mathrm{v}+\gamma-\beta'\mathrm{k}}\,\Gamma\left(1+\dfrac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\gamma+\mathrm{v}-\beta'\mathrm{k}}\,2^{\gamma+\mathrm{v}+\eta-\beta'\mathrm{k}}} H_{P,Q}^{M,N}\left[\frac{c\,x^{\delta}}{[2a(1-\alpha)]^{\delta}}\Bigg|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right]$$

$$\cdot\ {}_1\psi_2\left[-\frac{x^2}{4a^2(1-\alpha)^2}\Bigg|\begin{matrix}(\gamma+\mathrm{v}-\delta\xi-\beta'\mathrm{k},2)\\(\mathrm{v}+1,1),(1+\mathrm{v}+\gamma-\delta\xi-\beta'\mathrm{k}+\frac{\eta}{1-\alpha},2\end{matrix}\right]. \qquad (26)$$

Here ${}_p\psi_q$ denotes the generalized Wright hypergeometric function [18].

(D) Letting $n' \to 0$ in the results (21) through (23), we get the results (11), (20) and (23) obtained by Chaurasia and Gill in [2].

(E) Taking $n' \to 0$ and assigning some suitable values to H-function in Theorems 1,2 and 3, we get the results derived by Nair in [12].

(F) By giving suitable values to the parameters in the results obtained by Nair [12], the new, known, unknown and several other interesting results from our results can be obtained.

4. Conclusion

In this paper, we introduce a new fractional integration operator associated with the pathway model and pathway probability density. The object of the present paper is to study a pathway model and pathway probability density for certain products of special functions with general argument. The importance of our results lies in their manifold generality. In view of the generality of H-functions and general class of polynomials, on specializing the various parameters therein, we can obtained from our main result, several result containing remarkably wide variety of useful functions and their various special cases. Thus the main results presented in this article would at once yield a very large number of results containing a large variety of simpler special functions occurring scientific and technological fields.

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